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EXISTENCE OF GLOBAL ATTRACTOR FOR CAUCHY PROBLEMS

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Abstract. We investigate the existence of a global attractor for the Cauchy problem

$$x'(t) = Ax(t) + F(x(t)),\ x(0) = x_0 \in X$$

on a Banach space X according to the remark in You and Yuan's paper.

1. Introduction

The theory of global attractor is one of important object to describe long time dynamics of an infinite dimensional system. It is very useful to investigate the asymptotic behavior since it is valid for more general situations than those for stability.

Caraballo et al. [5] proved the existence of attractors for

$$\begin{cases} x'(t) & = f(t, x_t),\ t > \tau \\ x_\tau & = \phi \in C_\gamma, \end{cases} \tag{1.1}$$

where C_γ is the Banach space defined by

$$C_\gamma = \{x \in C((-\infty, 0], \mathbb{R}^n) :\ \exists \lim_{\theta \to -\infty} e^{\gamma\theta} x(\theta)\},$$

with the norm

$$||x|| = \sup_{-\infty < \theta \leq 0} e^{\gamma\theta} |x(\theta)|,\ \gamma > 0,$$

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and $C((\infty,0],\mathbb{R}^n)$ is the space of continuous functions from $(-\infty,0]$ to $\mathbb{R}^n$ with the sup-norm

$$||\phi|| = \sup_{-\infty<s\le 0} |\phi(s)|,$$

and define $x_t \in C((-\infty,0],\mathbb{R}^n)$ by $x_t(s) = x(t+s),\ s \in (-\infty,0]$, for a function x.

Zelatti [12] studied the existence of a global attractor under minimal requirements in terms of continuity of the multivalued semigroup of operators acting on complete metric spaces.

Bouzahir and Ezzinbi [3] investigated the existence of a global attractor for the equation

$$\begin{cases} x'(t) & = Ax(t) + F(x_t),\ t \ge 0, \\ x_0 & = \phi \in B, \end{cases} \tag{1.2}$$

where $A : D(A) \subset X \to X$ is a closed linear operator on a Banach space X and B is the phase space by Hale and Kato [7]. Also, Bouzahir et al. studied the existence of a global attractor for (1.2) under the initial condition $x_0 = \varphi \in C_\gamma$ in [2]. You and Yuan [11] investigated the existence of a global attractor for (1.2) under the initial condition $x_0 = \varphi \in C = C([-r,0],X), r > 0$.

In this paper, we will prove the existence of a global attractor for the equation without delay $(r = 0)$

$$\begin{cases} x'(t) & = Ax(t) + F(x(t)),\ t \ge 0, \\ x(0) & = x_0 \in X, \end{cases} \tag{1.3}$$

according to Remark 3.3 in [11].

2. Existence of global attractor

We consider the partial functional differential equation

$$\begin{cases} x'(t) & = Ax(t) + F(x_t),\ t \ge 0, \\ x_0 & = \phi \in C = C([-r,0],X), r > 0. \end{cases} \tag{2.1}$$

Here $C([-r,0],X)$ is the space of continuous functions from $[-r,0]$ into a Banach space X. We assume that the following.

(H1) The linear operator A is a Hille-Yosida operator, that is, there exist $M \ge 1$ and $\omega \in \mathbb{R}$ such that

(i) $(\omega,\infty) \subset \rho(A)$

(ii) $||(\lambda I - A)^{-n}|| \le \frac{M}{(\lambda-\omega)^n}$ for $\lambda > \omega$, where $\rho(A)$ is the resolvent set of A.

(H2) $F : C \to X$ is Lipschitz continuous, i.e., there exists $L > 0$ such that

$$||F(\phi_1) - F(\phi_2)|| \leq L||\phi_1 - \phi_2||, \phi_1, \phi_2 \in C.$$

The history function $x_t \in C$ is defined by

$$x_t(\theta) = x(t + \theta),\ -r \leq \theta \leq 0.$$

DEFINITION 2.1. A continuous function $x : [-r, T] \to X$, $T > 0$, is called an *integral solution* of (2.1) if

(i) $\int_0^t x(s)ds \in D(A), t \geq 0$.
(ii) $x(t) = \phi(0) + A \int_0^t x(s)ds + \int_0^t F(x_s)ds$.
(iii) $x_0 = \phi$.

DEFINITION 2.2. A family $\{S(t)\}_{t\geq 0}$ of linear operators on X is called an *integral semigroup* on X if

(i) $S(0) = 0$.
(ii) $t \longmapsto S(t)$ is strongly continuous.
(iii) $S(s)S(t) = \int_0^s [S(t + r) - S(r)]dr,\ t, s \geq 0$.

DEFINITION 2.3. An operator A is said to be a *generator* of $\{S(t)\}_{t\geq 0}$ if there exists $\omega \in \mathbb{R}$ such that

(i) $(\omega, \infty) \subset \varphi(A)$.
(ii) $R(\lambda, A) = (\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt,\ \lambda > \omega$.

Kellerman and Hieber [8] obtained the following.

LEMMA 2.4. *A is a generator of a locally Lipschitz continuous integrated semigroup if and only if A is a Hille-Yosida operator.*

Let A_0 be a part of A in $\overline{D(A)}$ defined by

$$A_0 = A \text{ on } D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}.$$

LEMMA 2.5. [10] *A_0 generates a C_0 semigroup $\{T_0(t)\}_{t\geq 0}$ on $\overline{D(A)}$.*

Note that

$$S(t)x = \lim_{\lambda\to\infty} \int_0^t T_0(s)\lambda(\lambda I - A)^{-1} x ds. \tag{2.2}$$

In view of (2.2), the integral solution of (2.1) can be defined as the following.

Definition 2.6. Let $T > 0$ and $\phi \in C$ with $\phi(0) \in \overline{D(A)}$. A function $x : [-r, T] \to X$ is called an *integral solution* of (2.1) if

$$x(t) = \begin{cases} T_0(t)\phi(0) + \lim_{\lambda\to\infty} \int_0^t T_0(t-\tau)\lambda(\lambda I - A)^{-1}F(x_\tau)d\tau, 0 \le t \le T \\ \\ \phi(t), -r \le t \le 0. \end{cases} \tag{2.3}$$

Adimy and Ezzinbi [1] obtained the existence of integral solution of (2.1).

Lemma 2.7. *Under the conditions (H1) and (H2), (2.1) has a unique integral solution* $x : [-r, \infty) \to X$.

Definition 2.8. A *solution operator* $U(t) : \widetilde{C} \to X$ is defined by

$$U(t)\phi = x_t(\cdot, \phi), t \ge 0,$$

where $\widetilde{C} = \{\phi \in C : \phi(0) \in \overline{D(A)}\}$ and $x(\cdot, \phi)$ is an integral solution of (2.1).

Lemma 2.9. [1] $\{U(t)\}_{t\ge0}$ *is a* C_0 *semigroup on* $\widetilde{C}$.

The following dynamical notions are from [6].

Definition 2.10. A set $A \subset X$ *attracts* a set B under $U(t)$ if

$$\mathrm{dist}(U(t)B, A) \to 0 \text{ as } t \to \infty.$$

Definition 2.11. A set $A \subset X$ is *invariant* under $U(t)$ if for all $x \in A$, the orbit $\gamma(x) = U(t)x$ is contained in A.

Definition 2.12. An invariant set $A \subset X$ is called a *global attractor* of a semigroup $\{U(t)\}_{t\ge0}$ if A is a maximal compact invariant set which attracts each bounded set $B \subset X$.

Definition 2.13. A semigroup $\{U(t)\}_{t\ge0}$ is said to be *point dissipative* if there exists a bounded set $B \subset X$ that attracts each point of X under $U(t)$.

You and Yuan [11] proved the existence of a global attractor of (2.1) by using the following.

Lemma 2.14. [2] *If there exists* $t_0 > 0$ *such that* $U(t)$ *is compact for* $t > t_0$, *and* $U(t)$ *is point dissipative, then there exists a nonempty global attractor in* X.

Now, we consider the equation ($r = 0$ in (2.1))

$$\begin{cases} x'(t) & = Ax(t) + F(x(t)),\ t \geq 0, \\ x(0) & = x_0 \in X. \end{cases} \tag{2.4}$$

with the assumptions

(H1) $A : D(A) \subset X \to X$ is a Hille-Yosida operator.

(H2) $F : X \to X$ is Lipshitz continuous with Lipschitz constant L. Then (2.4) has a unique integral solution $x : [0, \infty) \to X$ defined by

$$x(t) = T_0(t)x_0 + \lim_{\lambda\to\infty} \int_0^t T_0(t-\tau)\lambda(\lambda I - A)^{-1}F(x(\tau))d\tau, t \geq 0 \tag{2.5}$$

by (2.3). Define $U(t) : X \to X$ by

$$U(t)x_0 = x(t).$$

Then, by Lemma 2.9, $\{U(t)\}_{t\geq 0}$ is a C_0 semigroup on X. Furthermore, we assume that

(H3) $||T_0(t)|| \leq e^{-\alpha t}$ for some constant $\alpha > 0$.

(H4) $T_0(t), t \geq 0$, is compact.

We need the following generalized Gronwall's inequality to obtain the estimation of the integral solution for (2.4).

Lemma 2.15. [11] *If*

$$x(t) \leq h(t) + \int_{t_0}^t k(s)x(s)ds,\ t_0 \leq t < T,$$

where all the functions involved are continuous on $[t_0, T), T \leq \infty$, *and* $k(t) \geq 0$, *then* $x(t)$ *satisfies*

$$x(t) \leq h(t) + \int_{t_0}^t h(s)k(s)e^{\int_s^t k(u)du}ds,\ t_0 \leq t < T.$$

Theorem 2.16. *Assume that (H1)-(H3) hold true. Then we have*

$$||x(t)|| \leq \frac{c_1}{\alpha - L} + (||x(0)|| - \frac{c_1}{\alpha - L})e^{(L-\alpha)t},\ t \geq 0, \tag{2.6}$$

where $c_1 = ||F(0)||$.

Proof. Note that

$$\begin{aligned} ||F(x)|| &= ||F(x) - F(0) + F(0)|| \leq ||F(0)|| + ||F(x) - F(0)|| \\ &\leq c_1 + L||x||. \end{aligned}$$

Then, from (2.5), we have

$$\begin{aligned}
&||x(t)|| \\
&\le ||T_0(t)||||x_0|| + \lim_{\lambda\to\infty}\int_0^t ||T_0(t-\tau)||||\lambda(\lambda I - A)^{-1}||||F(x(\tau))||d\tau \\
&\le e^{-\alpha t}||x_0|| + \lim_{\lambda\to\infty}\int_0^t c_1 e^{-\alpha(t-\tau)}\frac{\lambda}{\lambda-\omega}d\tau \\
&\quad + \lim_{\lambda\to\infty}\int_0^t e^{-\alpha(t-\tau)}\frac{\lambda}{\lambda-\omega}L||x(\tau)||d\tau \\
&\le e^{-\alpha t}||x_0|| + c_1 e^{-\alpha t}\int_0^t e^{\alpha\tau}d\tau + Le^{-\alpha t}\int_0^t e^{\alpha\tau}||x(\tau)||d\tau \\
&\le e^{-\alpha t}[||x_0|| + \frac{c_1}{\alpha}(1-e^{-\alpha t})] + Le^{-\alpha t}\int_0^t e^{\alpha\tau}||x(\tau)||d\tau.
\end{aligned}$$

Thus

$$e^{\alpha t}||x(t)|| \le ||x_0|| + \frac{c_1}{\alpha}(e^{\alpha t}-1) + L\int_0^t e^{\alpha\tau}||x(\tau)||d\tau.$$

In view of the generalized Gronwall's inequality (Lemma 2.15) we have

$$\begin{aligned}
e^{\alpha t}||x(t)|| \quad &\le ||x_0|| + \frac{c_1}{\alpha}(e^{\alpha t}-1) + L\int_0^t [||x_0|| + \frac{c_1}{\alpha}(e^{\alpha s}-1)]e^{\int_s^t Ld\tau}ds \\
&= ||x_0|| + \frac{c_1}{\alpha}(e^{\alpha t}-1) + L||x_0||e^{Lt}\int_0^t e^{-Ls}ds \\
&\quad + \frac{c_1 L}{\alpha}e^{Lt}(\int_0^t e^{(\alpha-L)s}ds - \int_0^t e^{-Ls}ds) \\
&= ||x_0|| + \frac{c_1}{\alpha}(e^{\alpha t}-1) + ||x_0||(e^{Lt}-1) \\
&\quad + \frac{c_1 L}{\alpha(\alpha-L)}(e^{\alpha t}-e^{Lt}) - \frac{c_1}{\alpha}(e^{Lt}-1) \\
&= (\frac{c_1}{\alpha} + \frac{c_1 L}{\alpha(\alpha-L)})e^{\alpha t} + (||x_0|| - \frac{c_1 L}{\alpha(\alpha-L)} - \frac{c_1}{\alpha})e^{Lt} \\
&= \frac{c_1}{\alpha-L}e^{\alpha t} + (||x_0|| - \frac{c_1}{\alpha-L})e^{Lt},\ t\ge 0.
\end{aligned}$$

Hence we obtain the estimation (2.6). This completes the proof. □

Corollary 2.17. *Under the hypotheses of Theorem 2.16, $U(t), t \ge 0$, is point dissipative if $\alpha \ne L$.*

Proof. There exists $t_0 > 0$ such that

$$||x(t)|| \leq \frac{c_1}{\alpha - L} + 1 = k, t \geq t_0$$

by Theorem 2.16. It follows that the ball $B(0, k)$ attracts each point of X. □

THEOREM 2.18. *Assume that (H1)-(H4) hold true. Then $U(t)$ is compact for $t > 0$.*

Proof. Let (ϕ_n) be a bounded sequence in $\widetilde{C}(\mathbb{R}, X) = \{x \in C(\mathbb{R}, X) : x(0) \in \overline{D(A)}\}$. We show that $\{U(t)\phi_n : n \in \mathbb{N}\}$ is relatively compact in $\widetilde{C}(\mathbb{R}, X)$. Note that

$$\begin{aligned}
&(U(t)\phi_n)(\theta)\\
&= T_0(t+\theta)\phi_n(0) + \lim_{\lambda\to\infty}\int_0^{t+\theta} T_0(t+\theta-\tau)\lambda(\lambda I - A)^{-1}F(x(\tau))d\tau.
\end{aligned}$$

Since (ϕ_n) is bounded and $T_0(t)$ is compact by (H4), $\{T_0(t+\theta)\phi_n(0) : n \in \mathbb{N}\}$ is relatively compact. We have, for small $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{\lambda\to\infty}\int_0^{t+\theta} T_0(t+\theta-\tau)\lambda(\lambda I - A)^{-1}F(x(\tau))d\tau\\
&= T_0(\varepsilon)\lim_{\lambda\to\infty}\int_0^{t+\theta-\varepsilon} T_0(t+\theta-\tau-\varepsilon)\lambda(\lambda I - A)^{-1}F(x(\tau))d\tau\\
&\quad + \lim_{\lambda\to\infty}\int_{t+\theta-\varepsilon}^{t+\theta} T_0(t+\theta-\tau)\lambda(\lambda I - A)^{-1}F(x(\tau))d\tau.
\end{aligned}$$

Note that $\sup||x(t)|| < \infty$ by Theorem 2.16 and $||F(x)|| \leq ||F(0)|| + L||x||$ by (H2). Thus there exist $M_1 > 0$ and $M_2 > 0$ such that

$$||\lim_{\lambda\to\infty}\int_{t+\theta-\varepsilon}^{t+\theta} T_0(t+\theta-\tau)\lambda(\lambda I - A)^{-1}F(x(\tau))d\tau|| \leq M_2$$

and

$$||\lim_{\lambda\to\infty}\int_0^{t+\theta-\varepsilon} T_0(t+\theta-\tau-\varepsilon)\lambda(\lambda I - A)^{-1}F(x(\tau))d\tau|| \leq M_1\varepsilon.$$

This implies that

$$T_0(\varepsilon)\{\lim_{\lambda\to\infty}\int_0^{t+\theta-\varepsilon} T_0(t+\theta-\tau-\varepsilon)\lambda(\lambda I - A)^{-1}F(x(\tau))d\tau\} \subset K_\varepsilon$$

for some compact set $K_\varepsilon \subset X$. Therefore $\{(U(t)\phi_n)(\theta) : n \in \mathbb{N}\}$ is relatively compact. Also, we can show that the family $\{U(t)\phi_n : \phi_n \in$

$\widetilde{C}(\mathbb{R}, X)\}$ is equicontinuous. Therefore $U(t)$ is compact for $t > 0$. This completes the proof. $\square$

Now, we obtain the following result by Theorem 2.16, Corollary 2.17 and Theorem 2.18.

THEOREM 2.19. *Assume that (H1)-(H4) hold true. Then (2.4) has a nonempty global attractor provided* $\alpha > L$.

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